# IDEAL CLASS GROUP ANNIHILATORS

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Abstract. We study certain correspondences over Drinfeld modular varieties given by sums of Hecke correspondences. We propose generalizations of Stickelberger's theorem for higher dimensions. Using this result, we study anihilators for some cusp forms.

## Contents



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## 1. Introduction.

The aim of this article is to propose generalizations of Stickelberger's theorem for higher dimensions. Using these results, we study anihilators for some cusp forms. We address certain correspondences, given by sums of Hecke correspondences and defined over Drinfeld modular varieties .

Let $\mathbb{P}^1$ be the projective line scheme over $\mathbb{F}_q$, $\mathbb{P}^1 \setminus \{\infty\} = Spec(\mathbb{F}_q[t])$ and let $I = p(t)\mathbb{F}_q[t]$ be an ideal in $\mathbb{F}_q[t]$ with $deg(p(t)) = d+1$. There exists an abelian Galois extension, $K_I^{\infty}/\mathbb{F}_q(t)$, of group $G_I \simeq (\mathbb{F}_q[t]/I)^{\times}$. These fields are the Carlitz extensions and are the cyclotomic fields in the case of function fields. c.f. [Ca].

Let us consider the $S$-incomplete $L$-function evaluator ($S := |I| \cup \{\infty\}$)

$$\prod_{x \in |\mathbb{P}^1| \setminus S} (1 - \tau_x \cdot z^{deg(x)})^{-1},$$

$\tau_x \in G_I$ being the Frobenius element for $x \in |\mathbb{P}^1|$. This Euler product can be expressed as:

$$Q(z) + \frac{(\sum_{h \in G_I} h) \cdot z^{d+1}}{1 - q \cdot z},$$

$Q(z)$ being a polynomial in $\mathbb{Z}[G_I][z]$ of degree $d$. If one denotes $Q(z) := \sum_{i=0}^{d} \gamma_i \cdot z^i$, with $\gamma_i \in \mathbb{Z}[G_I]$, then the correspondence

$$\sum_{i=0}^{d} \Gamma(Fr^{d-i}) * \Gamma(\gamma_i)$$

is trivial on $Spec(K_I^{\infty} \otimes K_I^{\infty})$. This is proved for $S = \{0, 1, \infty\}$ in [C] and for general $S = \{?, \infty\}$ in [An1]. This result is analogous to the function field case of Stickelberger's theorem. Here, $\Gamma(Fr^i)$ denotes the graphic of the Frobenius morphism, $Fr^i$, and $\Gamma(\gamma_i)$ is a sum of graphics of elements of $G_I$. For arbitrary smooth curves analogous results can be found in [Al2].

These trivial correspondences give an annihilating polynomial for the operator given by the correspondence $\Gamma(Fr)$ acting on the $\mathbb{Q}[G_I]$-module, $H^1((Y_I^{\infty})_{\bar{\mathbb{F}}}, \mathbb{Q}_l)$, and this implies proofs of the Brumer-Stark conjecture in the function field case ([An1], [C], [H1], [Ta], [Al2]). $Y_I^{\infty}$ denotes the Riemann variety associated with $K_I^{\infty}/\mathbb{F}_q$.

Here, we study the Euler products

$$\prod_{x\in|\mathbb{P}^1|\setminus S}\frac{1}{1-\sigma_1^x\cdot z+q\sigma_2^x\cdot z^2-\cdots+(-1)^n q^{n(n-1)/2}\sigma_n^x\cdot z^n}=\sum_{m\geq 0}T(m)\cdot z^m,$$

where, "$T(m)$" and "$\sigma_j^x$" are Hecke correspondences over certain modular Drinfeld varieties of dimension $n$, $\mathcal{E}_{n,?}^{I\infty}$. For precision in the notation, see section 2.2.

We prove

**Theorem 1** The correspondence

$$T(n\cdot d)+\Gamma(Fr)*T(n\cdot d-1)+\cdots+\Gamma(Fr^{n\cdot d-1})*T(1)+\Gamma(Fr^{n\cdot d})$$

is trivial(=rationally equivalent to 0 as cycles) in $\mathcal{E}_{n,?}^{I\infty}\times\mathcal{E}_{n,?}^{I\infty}$.

$*$ denotes the product of correspondences. This result for $n=1$ gives us Stickelberger's theorem for the cyclotomic function fields, [An1]. To do so, we study the isogenies of Drinfeld modules, as stated in [Gr2].

$\mathcal{E}_{n,?}^{I\infty}$ are affine schemes over $Spec(\mathbb{F}_q[t,1/h(t)]$, and $h(t)$ is a polynomial which depends on $I$. We denote $\mathcal{E}_2(I\infty):=\mathcal{E}_{2,?}^{I\infty}\otimes_{\mathbb{F}_q[t]}\mathbb{F}_q(t)$. $\mathcal{E}(I\infty)$ is a smooth affine curve which is defined over $K_I^\infty$. $\overline{\mathcal{E}}(I\infty)$ denotes the projective curve over $K_I$ associated with $\mathcal{E}(I\infty)$. Theorem 1 has the following results as a sequel.

**Lemma** $T(2d)+T(2d-1)+\cdots+T(1)+\Gamma(Id)$ annihilates the group $Pic(\mathcal{E}(I\infty))$. It seems to be a Stickelberger's theorem for the affine modular curve $\mathcal{E}(I\infty)$ over $K_I^\infty$.

There exists an arithmetic subgroup, $\Gamma_{I\infty}$, of $Gl_2(\mathbb{F}_q[t])$ such that if we denote by $\Omega$ the Drinfeld's upper half-plane and by $\overline{M}_{\Gamma_{I\infty}}$ the smooth projective model of the algebraic curve associated with $\Omega/\Gamma_{I\infty}$, we have:

$$\overline{M}_{\Gamma_{I\infty}}=\overline{\mathcal{E}}(I\infty)\otimes_{K_I^\infty}C,$$

$C$ being the algebraic closure of the completion of $\mathbb{F}_q(t)$ at $\infty$. In the usual way cusp forms for $\Gamma_{I\infty}$ are given by $H^0(\overline{M}_{\Gamma_{I\infty}},\Omega_{\overline{M}_{\Gamma_{I\infty}}})$. Here we follow the notation and results of [GR]. For the definition and study of cusps forms, readers are referred to the works of Gekeler, Goss or the Habilatationshrift of Gebhard Böckle.

From the above lemma we obtain an additive version of Stickelberger's theorem for $n=2$:

**Theorem 2** If the cardinal of the group $Pic(\mathcal{E}(I\infty))$ is $\infty$, there exists a cusp form for $\Gamma_{I\infty}$ that is annihilated by $\tilde{T}(d\cdot 2)+\tilde{T}(d\cdot 2-1)+\cdots+\tilde{T}(1)+Id$.

$\tilde{T}(j)$ denotes the linear operator given by j-hecke acting on the cusp forms.

From theorem 1, we also obtain ideal class group annihilators for the cyclotomic function fields in the spirit of Stickelberger's theorem. We prove that the correspondence

$$\sum_{i=0}^{nd}[\Gamma(Frob^{nd-i})*(\sum_{\substack{q(t)\ \text{monic}\in\mathbb{F}_q[t]\\ (I,q(t))=1, deg(q(t))=i}}\varphi(q(t),n)\cdot\Gamma(q(t))]$$

is trivial on $Spec(K_I^\infty\otimes K_I^\infty)$. $\Gamma(q(t))$ denotes the graphic of the element of $G_I$ associated with the class of $q(t)$ in $(\mathbb{F}_q[t]/I)^\times$. $\varphi(q(t),n)$ indicates the number of submodules $N\subseteq\mathbb{F}_q[t]^{\oplus n}$ such that:

$$\mathbb{F}_q[t]^{\oplus n}/N\simeq\mathbb{F}_q[t]/q_1(t)\oplus\cdots\oplus\mathbb{F}_q[t]/q_n(t)$$

with the product of the invariant factors $q_1(t)\cdots q_n(t)$ equal to $q(t)$. This latter result can also obtained in a more direct way by using the Euler product of section 2.4 and this result for $n=1$. Bearing in mind the analogy between Drinfeld varieties in positive characteristic and modular curves for number fields, I believe that the interest of this work is the possible translation of results to modular curves.

**List of notations**

$\mathbb{F}_q$ is a finite field with $q$-elements, $(q=p^m)$.

$\otimes$ denotes $\otimes_{\mathbb{F}_q}$.

$\mathcal{O}_{\mathbb{P}^1}$ denotes the ring sheaf of the scheme $\mathbb{P}^1$.

$R$ is an $\mathbb{F}_q$-algebra.

$R^\times$ denotes the group of units in a ring $R$. $F$, denotes the Frobenius morphism.

If $s\in Spec(R)$, then we denote by $k(s)$ the residual field associate with $s$.

$\mathbb{P}^1_R$ denotes $\mathbb{P}^1 \otimes R$.

Let $S$ be a finite subset of geometric points of $\mathbb{P}^1$. We denote by $\mathbb{A}^S$ the adele group outside $S$, and $O^S$ denotes the adeles within $\mathbb{A}^S$ without poles.

Let $M$ be a vector bundle over $\mathbb{P}^1$; $M(k)$ denotes $M \otimes_{\mathcal{O}_{\mathbb{P}^1}} \mathcal{O}_{\mathbb{P}^1}(k\infty)$, $k \in \mathbb{Z}$.

If $f : X \to X$ is a morphism of separated schemes, then $\Gamma(f)$ denotes the graphic of $f$; $\Gamma(f) = \{(f(x), x); \quad x \in X\}$.

$|\mathbb{P}^1|$ and $|I|$ denote the geometric points of $\mathbb{P}^1$ and $Spec(\mathbb{F}_q[t]/I)$, respectively.

## 2. Elliptic sheaves and Hecke correspondences. Euler products

In this section, except for proposition 2.3, all results are valid for any smooth, geometrically irreducible curve over $\mathbb{F}_q$ provided with a rational point $\infty$, although we only consider the projective line curve.

2.1. **Elliptic sheaves.** In this section we recall the definition of elliptic sheaves and level structures over an ideal $I \subset \mathbb{F}_q[t]$, [BlSt], [Dr2], [LRSt], [Mu].

**Definition 2.1.** *An elliptic sheaf of rank $n$ over $R$, $E := (E_j, i_j, \tau)$, is a commutative diagram of vector bundles of rank $n$ over $\mathbb{P}^1_R$, and injective morphisms of modules $\{i_h\}_{h\in\mathbb{N}}$, $\tau$:*

$$\begin{array}{ccccccccc}
E_1 & \xrightarrow{i_1} & E_2 & \xrightarrow{i_2} & \cdots & \xrightarrow{i_{(n-1)}} & E_n & \xrightarrow{i_n} & \cdots \\
\uparrow{\scriptstyle\tau} & & \uparrow{\scriptstyle\tau} & & \uparrow{\scriptstyle\tau} & & \uparrow{\scriptstyle\tau} & & \\
E_0^\sigma & \xrightarrow{i_0^\sigma} & E_1^\sigma & \xrightarrow{i_1^\sigma} & \cdots & \xrightarrow{i_n^\sigma} & E_n^\sigma & \longrightarrow & \cdots
\end{array}$$

*(here, $E_i^\sigma$ denotes $(Id \times F)^* E_i$), satisfying:*

*a) For any $s \in Spec(R)$, we fix $deg((E_j)_s) = j$.*

*b) For all $j \in \mathbb{Z}$, $E_{j+n} = E_j(1)$. We can assume that the $i_k$ are natural inclusions.*

*c) $E_j + \tau(E_j^\sigma) = E_{j+1}$ for all $j$.*

*d)* $\alpha^*(E_i/E_{i-1})$ *is a rank-one free module over* $R$*,* $\alpha$ *being the natural inclusion* $\infty \times Spec(R) \hookrightarrow \mathbb{P}^1_R$*.*

*Remark* 1. From these properties, it may be deduced that $h^0(E_j) = n + j$ and $h^1(E_j) = 0$, $j \geq -n$, c.f. [BlSt], [Dr2].

Moreover, it is seen that for the $R$-module $H^0(\mathbb{P}^1_R, E_j)$, $(j > -n)$, there exists a basis $\{s, \tau s, \cdots, \tau^{n+j-1}s\}$ with $\tau s := \tau((Id \times F)^* s)$ and $\tau^h s := \tau((Id \times F)^* \tau^{h-1} \cdot s)$.

**Definition 2.2.** *An* $I$*-level structure,* $\iota_I$*, for the elliptic sheaf* $(E_j, i_j, \tau)$ *is an* $I$*-level structure,* $\iota_{j,I}$*, for each vector bundle* $E_j$ *compatible with the morphisms* $\{i_j, \tau\}$*. i.e.,* $(\iota_{j+1,I}).i_j = \iota_{j,I}$ *and* $(\iota_{j+1,I}) \cdot \tau = (Id \times F)^*(\iota_{j,I})$*. We denote by* $(E, \iota_I)$ *an elliptic sheaf with an* $I$*-level structure.*

Recall that an $I$-level structure for a vector bundle $E_j$ over $\mathbb{P}^1_R$ is a surjective morphism of modules $E_j \to (\beta_*(R[t]/I))^{\oplus n}$, where $\beta : Spec(R[t]/I) \hookrightarrow \mathbb{P}^1_R$ is the natural inclusion.

The elliptic sheaf $(E_j, i_j, \tau)$ defined over $R$ gives a $\tau$-sheaf, $R\{\tau\} = \oplus_{i=0}^{\infty} R \cdot \tau^i$, $(\tau \cdot b = b^q \cdot \tau)$. One can identify:

$$H^0(\mathbb{P}^1_R, E_j) = \oplus_{i=0}^{n+j-1} R \cdot \tau^i s,$$

and in this way $R\{\tau\}$ is isomorphic to the graded $R[t]$-module:

$$\cup_{i=0}^{\infty} H^0(\mathbb{P}^1_R, E_j(i)).$$

*Remark* 2. By taking the determinant of $(E, \iota_I)$ we obtain an elliptic sheaf of rank 1, $(det(E_j), det(i_j), det(\tau))$, with an $I$-level structure $det(\iota_I)$. This determinant is studied in detail in [Ge].

The $\tau$-sheaf associated with $(det(E_j), det(i_j), \tau_{det})$ is

$$R\{\tau_{det}\} := \oplus_{i=0}^{\infty} R \cdot \tau^i_{det},$$

with $\tau^i_{det} = \tau^i \wedge \tau^{i+1} \wedge \cdots \wedge \tau^{n+i-1}$, $(i \geq 0)$. Moreover, $\wedge^n R\{\tau\} = R\{\tau_{det}\}$ as $R[t]$-modules.

We denote $det(E_j)$ by $L_j$; so $deg(L_j) = j$; recall that $deg(E_j) = j$.

**Proposition 2.3.** With the above notations, if $r_n - r_1 \geq n$ then:

$$\tau^{r_1} \wedge \tau^{r_2} \wedge \cdots \wedge \tau^{r_n} \in H^0(\mathbb{P}^1_R, L_{r_n - n}).$$

*Proof.* Since

$$\tau_{det}^{r_1}(1 \wedge \tau^{r_2 - r_1} \wedge \cdots \wedge \tau^{r_n - r_1}) = \tau^{r_1} \wedge \tau^{r_2} \wedge \cdots \wedge \tau^{r_n} \in H^0(\mathbb{P}^1_R, L_{r_n - n}), \quad (0 \leq r_1 \leq \cdots \leq r_n)$$

it suffices to prove the result for $r_1 = 0$.

We proceed by induction over $r_n$. For $r_n = n$, we have to prove that

$$1 \wedge \tau^{r_2} \wedge \cdots \wedge \tau^n \in H^0(\mathbb{P}^1_R, L_0),$$

$t \cdot a_n = a_0 + a_1 \cdot \tau + \cdots + \tau^n$ for $a_i \in R$, and therefore there exists $c \in R$ with

$$1 \wedge \tau^{r_2} \wedge \cdots \wedge \tau^n = c \cdot (1 \wedge \tau^2 \wedge \cdots \wedge \tau^{n-1}).$$

Recall that $(0 \leq r_2 \leq \cdots \leq n)$ and we conclude since

$$1 \wedge \tau^2 \wedge \cdots \wedge \tau^{n-1} \in H^0(\mathbb{P}^1_R, L_0).$$

Let us now assume the theorem is true for $k < r_n$. Let take us $r_n = l + n$. Thus,

$$1 \wedge \tau^{r_2} \wedge \cdots \wedge \tau^{r_n} = 1 \wedge \cdots \wedge \tau^{r_{n-1}} \wedge (t \cdot a_n \tau^{r_n - n} - a_0 \tau^{r_n - n} + a_1 \cdot \tau^{r_n - n + 1} + \cdots + a_{n-1} \tau^{r_n - 1}).$$

Since

$$1 \wedge \cdots \wedge \tau^{r_{n-1}} \wedge \tau^{r_n - i} \in H^0(\mathbb{P}^1_R, L_{r_n - n}), \text{ with } i \geq 1,$$

because

$$1, \cdots, \tau^{r_{n-1}}, \tau^{r_n - i} \in H^0(\mathbb{P}^1_R, E_{r_n - n}),$$

it suffices to prove that

$$(t \cdot a_n - a_0) \cdot (1 \wedge \cdots \wedge \tau^{r_{n-1}} \wedge \tau^{r_n - n}) \in H^0(\mathbb{P}^1_R, L_{r_n - n}).$$

Let consider us $k := max\{r_n - n, r_{n-1}\}$. If $n \leq k$ then we finish by induction, because $k + 1 \leq r_n$. When $k \leq n - 1$, it suffices to prove that

$$(t \cdot a_n - a_0) \cdot (1 \wedge \cdots \wedge \tau^{n-2} \wedge \tau^{n-1}) \in H^0(\mathbb{P}^1_R, L_{r_n - n}).$$

This is true because we are in the case $r_n - n \geq 1$.

□

2.2. **$\infty$-Level structures.** We shall now define level structures at $\infty \in \mathbb{P}^1$ over elliptic sheaves of rank 1. To do so, we take into account the results of [An1] 6.1.1. We take $t^{-1}$ as a local uniformizer at $\infty$.

The composition of the epimorphism

$$\mathcal{O}_{\mathbb{P}^1}(k) \to \mathcal{O}_{\mathbb{P}^1}(k)/\mathcal{O}_{\mathbb{P}^1}(k-1)$$

with the isomorphism induced by the multiplication by $t^{1-k}$

$$\mathcal{O}_{\mathbb{P}^1}(k)/\mathcal{O}_{\mathbb{P}^1}(k-1) \simeq \mathcal{O}_{\mathbb{P}^1}/\mathcal{O}_{\mathbb{P}^1}(-1),$$

gives us an $\infty$-level structure over $\mathcal{O}_{\mathbb{P}^1}(k)$.

**Definition 2.4.** *An $\infty$-level structure for a rank-1 elliptic sheaf, $(L_j, i_j, \tau)$, over $R$ is an $\infty$-level structure $(L_0, \iota_\infty)$ such that the diagram*

$$\begin{tikzcd} (*)\ L_0^\sigma \arrow[r, "\tau"] \arrow[dr, "\iota_\infty^\sigma"'] & L_0(1) \arrow[d, "t^{-1}\cdot\iota_\infty"] \\ & \gamma_*(R[t^{-1}]/t^{-1}R[t^{-1}]) \end{tikzcd}$$

*is commutative. Here, $\gamma : Spec(R[t^{-1}]/t^{-1}R[t^{-1}]) \hookrightarrow \mathbb{P}^1_R$, is the natural inclusion.*

We denote by, $\mathcal{E}_n^I$ and $\mathcal{E}_n^{I\infty}$ the moduli of elliptic sheaves with $I$-level structures, $(E, \iota_I)$, and with $I+\infty$-level structures, respectively. Where to give an $\infty$-level structure for $E$, $\iota_\infty$, is to give an $\infty$-level structure for the rank-1-elliptic sheaf $det(E)$. Henceforth, we denote with $(E, \iota_{I\infty})$ an element $(E, \iota_I, \iota_\infty) \in \mathcal{E}_n^{I\infty}$. There exists a morphism, $z : \mathcal{E}_n^{I\infty} \to Spec(\mathbb{F}_q[t])$, called the zero morphism, that is defined by:

$$z(E, \iota_{I\infty}) = supp(E_0/\tau(E_{-1}^\sigma)) = supp(det(E_0)/\tau_{det}(det(E_{-1})^\sigma).$$

Bearing in mind the antiequivalence between elliptic sheaves and Drinfeld modules, one can construct a ring $\mathcal{B}_n^I$ of dimension $n$ such that $Spec(\mathcal{B}_n^I) = \mathcal{E}_n^I$. For these results see [Dr1], [Dr2], [Lm], [Mu].

For $n = 1$, it is not hard to obtain a ring $\mathcal{B}_1^{I\infty}$, such that

$$Spec(\mathcal{B}_1^{I\infty}) = \mathcal{E}_1^{I\infty}.$$

Moreover, the morphism of forgetting the $\infty$-level structure

$$\mathcal{E}_1^{I\infty} \to \mathcal{E}_1^{I}$$

is étale, outside $|I|$. In the following remark we calculate $\mathcal{B}_1^{I\infty}$, explicitly.

*Remark* 3. We consider the rank-1-Drinfeld module $\phi_t = a\tau + \bar{t}$, defined over $\mathbb{F}_q[\bar{t}, a]$. We shall now study what an $\infty$-level structure for the Drinfeld module $\phi$ is.

Let us consider a rank-1 elliptic sheaf, $(L_j, i_j, \tau)$, associated with $\phi$ and let $\iota_\infty$ be an $\infty$-level structure for $(L_j, i_j, \tau)$. We have the morphisms of modules:

$$\iota_\infty : L_0 \to \mathbb{F}_q[\bar{t}, a][t^{-1}]/t^{-1}\mathbb{F}_q[\bar{t}, a].$$

We choose $s$ with $H^0(L_0) =< s >$. Note that $s$ is a generator of the line bundle $L_0$. We set $\iota_\infty(s) = \lambda$, and hence

$$\iota_\infty^\sigma : L_0^\sigma \to \mathbb{F}_q[a, \bar{t}][t^{-1}]/t^{-1}\mathbb{F}_q[a, \bar{t}]$$

gives $\iota_\infty^\sigma(s) = \lambda^q$. Also,

$$t^{-1} \cdot \iota_\infty : L_0(1) \to \mathbb{F}_q[\bar{t}, a][t^{-1}]/t^{-1}\mathbb{F}_q[a, \bar{t}]$$

is such that $t^{-1} \cdot \iota_\infty(\tau(s)) = t^{-1}\tau s = a^{-1}$, because $t \cdot s = a \cdot \tau s + \bar{t} \cdot s$ and $\bar{t} \cdot t^{-1} = 0$ as element in

$$\mathbb{F}_q[a, \bar{t}][t^{-1}]/t^{-1}\mathbb{F}_q[a, \bar{t}].$$

Therefore, the above diagram is commutative if and only if $\lambda^q = \lambda \cdot a^{-1}$. Thus, we can choose $\bar{s} := \lambda \cdot s \in H^0(M_0)$ such that $\iota_\infty(\bar{s}) = 1$. Therefore $t \cdot \bar{s} = \tau s + \bar{t}$, and we obtain the Drinfeld module $\bar{\phi}_t = \tau + \bar{t}$ isomorphic to $\phi_t$. It is not hard to see that $\mathcal{B}_1^{\infty} = \mathbb{F}_q[\bar{t}]$.

We set $I = p(t)\mathbb{F}_q[t]$, where $d+1$ is the degree of $p(t)$.

$$\bar{\phi}_{p(t)} = c_{d+1}\tau^{d+1} + \cdots + c_1\tau + p(\bar{t}); \quad c_i \in \mathbb{F}_q[\bar{t}].$$

We have that $\mathcal{B}_1^{I\infty} = \mathbb{F}_q[\bar{t}, p(\bar{t})^{-1}, \delta]$ with $\delta$ an element of a closed field of $\mathbb{F}_q(t)$ verifying:

$$\phi_{p(t)}(\delta) = \delta^{q^{d+1}} + \cdots + c_1\delta^q + r(\bar{t})\delta = 0,$$

and $\phi_{h(t)}(\delta) \neq 0$ with $h(t)$ a proper divisor of $r(t)$. The $I$-level structure for $(L_j, i_j, \tau)$ is given by

$$\iota_I(\bar{s}) = \bar{\phi}_{t^{r-1}}(\delta) + \bar{\phi}_{t^d}(\delta)t + \cdots + \bar{\phi}_{t^0}(\delta)t^d \in \mathbb{F}_q[\bar{t}, \delta][t]/p(t).$$

The morphism $\mathbb{F}_q[t] \hookrightarrow \mathcal{B}_1^{I\infty}$ $(t \to \bar{t})$ gives us the Galois extension $K_I/\mathbb{F}_q(t)$ of group $(\mathbb{F}_q[t]/I)^\times$.

By considering

$$det(E, \iota_I) := (det(E), det(\iota_I)),$$

and the determinant morphism

$$det : \mathcal{E}_n^I \to \mathcal{E}_1^I,$$

we obtain:

$$\mathcal{E}_n^{I\infty} = \mathcal{E}_n^I \times_{\mathcal{E}_1^I} \mathcal{E}_1^{I\infty},$$

and therefore, $\mathcal{E}_n^{I\infty}$ is an affine scheme of finite type over $\mathbb{F}_q$. It is smooth because the projection

$$\mathcal{E}_n^I \times_{\mathcal{E}_1^I} \mathcal{E}_1^{I\infty} \to \mathcal{E}_n^I$$

is étale since $\mathcal{E}_1^{I\infty} \to \mathcal{E}_1^I$ is also étale . Note that $\mathcal{E}_1^{I\infty}$ is defined over $\mathbb{P}^1 \setminus |I| \cup \infty$.

2.3. **Hecke correspondences.** We consider, $J_1 \subseteq \cdots \subseteq J_n$, a chain of ideals of $\mathbb{F}_q[t]$ coprime to $I$ and $S = |I| \cup \{\infty\}$.

Let $(E, \iota_{I\infty})$ be an elliptic sheaf defined over $R$ with level structures on $I$ and on $\infty$ and with zero outside $|J_1|$. We denote by $\mathcal{E}_{n,|J_1|}^{I\infty}$ the moduli scheme

$$\mathcal{E}_n^{I\infty} \times_{\mathbb{P}^1} (\mathbb{P}^1 \setminus |J_1|),$$

where the fibred product is obtained from the zero morphism $z : \mathcal{E}_n^{I\infty} \to \mathbb{P}^1$ and the natural inclusion $\mathbb{P}^1 \setminus |J_1| \hookrightarrow \mathbb{P}^1$.

We denote by

$$T(J_1, \cdots, J_n) \subset \mathcal{E}_{n,|J_1|}^{I\infty} \times \mathcal{E}_{n,|J_1|}^{I\infty}$$

the Hecke correspondence, which is given by the pairs

$$[(E, \iota_{I\infty}), (\bar{E}, \bar{\iota}_{I\infty})] \in \mathcal{E}_{n,|J_1|}^{I\infty} \times \mathcal{E}_{n,|J_1|}^{I\infty},$$

$E$ being a sub-elliptic sheaf of $\bar{E}$ such that for each $s \in Spec(R)$ we have

$$\bar{E}_s/E_s \simeq k(s)[t]/J_1 \oplus \cdots \oplus k(s)[t]/J_n.$$

The $I+\infty$-level structure, $\iota_{I\infty}$, defined over $E$ is the composition $\bar{\iota}_{I\infty} \cdot \rho$, $\rho$ being the inclusion $E \subset \bar{E}$.

We shall now describe the Hecke correspondences in an adelic way. To do so, consider $(\bar{E}, \bar{\iota}_{I\infty})$ defined over an algebraic closed field $K$.

We denote by

$$\pi_1, \pi_2 : \mathcal{E}^{I\infty}_{n,|J_1|} \times \mathcal{E}^{I\infty}_{n,|J_1|} \to \mathcal{E}^{I\infty}_{n,|J_1|}$$

the natural projections. There exists a bijection between the sets:

$$\pi_1(\pi_2^{-1}(\bar{E}, \bar{\iota}_{I\infty}) \cap T(J_1, \cdots, J_n)), \quad \pi_2(\pi_1^{-1}(\bar{E}, \bar{\iota}_{I\infty}) \cap T(J_1, \cdots, J_n)),$$

and the $\mathbb{F}_q[t]$-submodules $M$ and $\bar{M}$,

$$M \subseteq \mathbb{F}_q[t]^{\oplus n} \subseteq \bar{M}$$

with

$$\mathbb{F}_q[t]^{\oplus n}/M \simeq \bar{M}/\mathbb{F}_q[t]^{\oplus n} \simeq \mathbb{F}_q[t]/J_1 \oplus \cdots \oplus \mathbb{F}_q[t]/J_n$$

respectively. These sets have the same cardinal, which we denote by $d(J_1, \cdots, J_n)$.

In the following proposition, $\mathcal{C}ht^I_{n,|J_1|}$, denotes the stack of shtuckas of rank $n$ with zeroes outside $|J_1|$ and level structures over $I$. c.f. [Lf]

**Proposition 2.5.** $T(J_1, \cdots, J_n)$ is a closed subscheme of $\mathcal{E}^{I\infty}_{n,|J_1|} \times \mathcal{E}^{I\infty}_{n,|J_1|}$. Moreover, the morphisms $\pi_1, \pi_2$ restricted to $T(J_1, \cdots, J_n)$ are étale morphisms. We denote these morphisms by $\bar{\pi}_1, \bar{\pi}_2$, respectively.

*Proof.* Let consider us the morphism:

$$\mathfrak{e} : \mathcal{E}^I_{n,|J_1|} \to \mathcal{C}ht^I_{n,|J_1|},$$

defined by

$$\mathfrak{e}(E, \iota_I) := (( E_{-1} \xrightarrow{i} E_0 \xleftarrow{\tau} E^{\sigma}_{-1} ), \iota_I).$$

c.f. [Dr3], (pag.109).

The Hecke correspondences $\Gamma^n(g)$ defined in [Lf] (section I, 4) are closed substacks within $\mathcal{C}ht^I_{n,|J_1|} \times \mathcal{C}ht^I_{n,|J_1|}$. Let us consider $g \in Gl_n(\mathbb{A}^S)$ such that

$$\oplus^n O^S / g(\oplus^n O^S) \simeq \mathbb{F}_q[t]/J_1 \oplus \cdots \oplus \mathbb{F}_q[t]/J_n$$

as modules. In this way,

$$T^I(J_1, \cdots, J_n) := (\mathfrak{e} \times \mathfrak{e})^* \Gamma^n(g)$$

is a closed subscheme of $\mathcal{E}^I_n \times \mathcal{E}^I_n$, where $T^I(J_1, \cdots, J_n)$ denotes the Hecke correspondence

$$(\pi_\infty \times \pi_\infty)(T(J_1, \cdots, J_n)) \subset \mathcal{E}^I_{n,|J_1|} \times \mathcal{E}^I_{n,|J_1|},$$

$\pi_\infty : \mathcal{E}^{I\infty}_{n,|J_1|} \to \mathcal{E}^I_{n,|J_1|}$ being the morphism of forgetting the $\infty$-level structure. Now, $T(J_1, \cdots, J_n)$ is the closed subscheme given by the pairs:

$$[(E, \iota_{I\infty}), (\bar{E}, \bar{\iota}_{I\infty})] \in (\pi_\infty \times \pi_\infty)^{-1} T^I(J_1, \cdots, J_n),$$

such that

$$\begin{tikzcd} det(E) \arrow[r, hook, "det(\rho)"] \arrow[dr, "\iota_\infty"'] & det(\bar{E}) \arrow[d, "\bar{\iota}_\infty"] \\ & \gamma_*(R[t^{-1}]/t^{-1}R[t^{-1}]) \end{tikzcd}$$

is commutative. $det(\rho) : det(E) \hookrightarrow det(\bar{E})$ is the determinant of the injective morphism given between the elliptic sheaves $\rho : E \hookrightarrow \bar{E}$.

Because

$$T^I(J_1, \cdots, J_n) = \Gamma^n(g) \times_{\mathcal{C}ht^I_{n,|J_1|}} \mathcal{E}^I_{n,|J_1|},$$

and since the projections $p_i : \Gamma^n(g) \to \mathcal{C}ht^I_{n,|J_1|}$, $(i = 1, 2)$ are étale morphisms, we have that the two projections from $T^I(J_1, \cdots, J_n)$ to $\mathcal{E}^I_{n,|J_1|}$ are étale morphisms.

We conclude that $\bar{\pi}_1, \bar{\pi}_2$ are étale morphisms because

$$T(J_1, \cdots, J_n) = T^I(J_1, \cdots, J_n) \times_{\mathcal{E}^I_{n,|J_1|}} \mathcal{E}^{I\infty}_{n,|J_1|}.$$

They are morphisms of degree $d(J_1, \cdots, J_n)$.

$\square$

The formal sum of Hecke correspondences gives a commutative ring where the product is the composition of correspondences. This ring is isomorphic to the

commutative ring

$$C_c(K\backslash Gl_n(\mathbb{A}^S)/K)$$

of $\mathbb{Z}$-valued continuous functions over $Gl_n(\mathbb{A}^S)$, invariant by the action of $K := Gl_n(O^S)$ on the left and on the right over $Gl_n(\mathbb{A}^S)$ and with compact support. The product is the convolution product. This isomorphism sends the correspondence $T(J_1, \cdots, J_n)$ to the characteristic function over the open compact subset:

$$Gl_n(O^S) \cdot (\mu_{J_1}, \cdots, \mu_{J_n}) \cdot Gl_n(O^S),$$

with $\mu_{J_i} \in \mathbb{A}^S$, given by the element $q_i(t)$ with $J_i = q_i(t)\mathbb{F}_q[t]$, and $(\mu_{J_1}, \cdots, \mu_{J_n})$ the diagonal matrix in $Gl_n(\mathbb{A}^S)$, where the diagonal is given by $(\mu_{J_1}, \cdots, \mu_{J_n})$.

We denote by $T(m)$ the correspondence defined by the formal sum of the Hecke correspondences $T(J_1, \cdots, J_n)$, where $\sum_{i=0}^n dim_{\mathbb{F}_q} \mathbb{F}_q[t]/J_i = m$.

As in the number field case, one can consider Hecke correspondences as operators over the abelian group of formal sums of $\mathbb{F}_q[t]$-submodules, $N$, of rank $n$ of $\mathbb{F}_q(t)^{\oplus n}$ (=lattices of $\mathbb{F}_q(t)^{\oplus n}$). One defines:

$$T(J_1, \cdots, J_n)(N) = \sum \bar{N},$$

where $\bar{N}$ runs over the submodules of $N$, satisfying:

$$N/\bar{N} \simeq \mathbb{F}_q[t]/J_1 \oplus \cdots \oplus \mathbb{F}_q[t]/J_n.$$

In this way $T(m)(N) = \sum_{\bar{N} \subset N} \bar{N}$, where $dim_{\mathbb{F}_q} N/\bar{N} = m$.

A more rigorous presentation of this section can be found in [Lf] and [Lm].

2.4. **Euler products.** A generalization for the non-abelian case of the $S$-incomplete $L$-function evaluator at $s = 0$, (cf: [H1], [Ta]) is studied in [H2]. In this section we address the issue in another way.

$\mathcal{E}^{I\infty}_{n,|\mathbb{P}^1|}$ denotes the moduli scheme of elliptic sheaves of rank $n$ with level structures over $I$ and $\infty$ and with zero outside $|\mathbb{P}^1|$.

Let $x \in |\mathbb{P}^1| \setminus S$, $(S = |I| \cup \{\infty\})$ and let $t_x$ be a local uniformizer for $x$. We consider the diagonal matrix

$$(\mu_x, \overset{j}{\overbrace{\cdots}}, \mu_x, 1, \cdots, 1) \in Gl_n(\mathbb{A}^S),$$

$\mu_x$ being the adele within $\mathbb{A}^S$ such that it is 1 over each place of $|\mathbb{P}^1| \setminus S \cup \{x\}$ and $t_x$ over $x$. We denote by $\sigma_j^x$, $1 \leq j \leq n$, the Hecke correspondence over $\mathcal{E}^{I\infty}_{n,|\mathbb{P}^1|}$ given by the characteristic function of

$$Gl_n(O^S) \cdot (\mu_x, \overset{j}{\smile\!\!\!\!\!\cdots}, \mu_x, 1, \cdots, 1) \cdot Gl_n(O^S).$$

In the following lemma, for easy notation we assume that $deg(x) = 1$ and $t_x$ is a local parameter for $x$. $\mathfrak{m}_x$ is the maximal ideal associated with $x$.

One can find a proof of the next lemma in, [Sh] Th 3.21. More or less, we repeat that proof.

**Lemma 2.6.** We have:

$$\frac{1}{1 - \sigma_1^x \cdot z + q\sigma_2^x \cdot z^2 - q^3\sigma_3^x \cdot z^3 + \cdots + (-1)^n q^{n(n-1)/2}\sigma_n^x \cdot z^n} = \sum_{m \geq 0} T^x(m) \cdot z^m,$$

where

$$T^x(m) \subset \mathcal{E}^{I\infty}_{n,|\mathbb{P}^1|} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}^1|}$$

denotes sum of the Hecke correspondences $T(\mathfrak{m}_x^{r_1}, \cdots, \mathfrak{m}_x^{r_n})$, $(r_1 \geq \cdots \geq r_n \geq 0)$, and $r_1 + \cdots + r_n = m$.

*Proof.* We shall model this proof as in [Ln]. It suffices to prove that for each $r \in \mathbb{N}$ we have "Newton's" formulas

$$P := T^x(r) - T^x(r-1) \cdot \sigma_1^x + q\sigma_2^x \cdot T^x(r-2) - \cdots + (-1)^n q^{n(n-1)/2} T^x(r-n) \cdot \sigma_n^x = 0$$

by denoting $T^x(0) = 1$ and $T^x(l) = 0$ for $l < 0$.

To accomplish this, we consider Hecke correspondences as operators over the formal abelian group of lattices, $\bar{N}$ and $N$ being lattices with $\bar{N} \subseteq N$, $dim_{\mathbb{F}_q} N/\bar{N} = r$ and $N/\bar{N}$ concentrated over $x$. We shall prove that the multiplicity of $\bar{N}$ in the formal sum $P(N)$ is 0.

$\sigma_j^x(N) = \sum N'$, where $N'$ belong to the sublattices $N' \subset N$ with

$$N/N' \simeq \mathbb{F}_q[t]/\mathfrak{m}_x \oplus \overset{j}{\smile\!\!\!\!\!\cdots} \oplus \mathbb{F}_q[t]/\mathfrak{m}_x$$

or, equivalently, the vector subspaces, $N'$ of codimension $j$, of $N/\mathfrak{m}_x \cdot N$.

If we denote $h := dim_{\mathbb{F}_q} N/(\mathfrak{m}_x \cdot N + \bar{N})$, then the number of lattices, "$N'$" such that $\bar{N} \subset N'$, is given by the number of $\mathbb{F}_q$-subvector spaces in $N/(\mathfrak{m}_x \cdot N + \bar{N})$ of codimension $j$. This number is given by the $q$-combinatorial number

$$\binom{h}{j}_q =: \frac{(q^h - 1) \cdots (q^{h-j+1} - 1)}{(q^j - 1) \cdots (q - 1)}$$

for $j \leq h$, and $\binom{h}{j}_q =: 0$ for either $j < 0$ or $j > h$.

We conclude the lemma bearing in mind the relation

$$\binom{h}{h}_q - \binom{h}{h-1}_q + q \cdot \binom{h}{h-2}_q - \cdots + (-1)^n q^{n(n-1)/2} \cdot \binom{h}{h-n}_q = 0.$$

I have taken this formula from Appendix D, [Lm], (cf. [Ma]).

□

**Theorem 2.7.** If we denote

$$L^x := \frac{1}{1 - \sigma_1^x \cdot z + q\sigma_2^x \cdot z^2 - q^3 \sigma_3^x \cdot z^3 + \cdots + (-1)^n q^{n(n-1)/2} \sigma_n^x \cdot z^n},$$

then

$$\prod_{x \in |\mathbb{P}^1| \setminus S} L^x = \sum_{m \geq 0} T(m) \cdot z^m.$$

*Proof.* It suffices to bear in mind the above lemma and that if $J_1$ and $\bar{J}_1$ are ideals coprime within $A$, then:

$$T(J_1, \cdots, J_n) \cdot T(\bar{J}_1, \cdots, \bar{J}_n) = T(J_1 \cdot \bar{J}_1, \cdots, J_n \cdot \bar{J}_n).$$

□

## 3. Isogenies and Hecke correspondences

Here, we study the isogenies between Drinfeld modules(=elliptic sheaves), [Gr2], to establish the relation between the above Euler products and isogenies between elliptic sheaves.

### 3.1. **Isogenies for elliptic sheaves.**

**Definition 3.1.** *An isogeny, $\Phi$, of degree $m \in \mathbb{N}$ between two elliptic sheaves with $I$-level structures $(E, \iota_{I\infty}), (\bar{E}, \bar{\iota}_{I\infty})$ and $\infty$-level structures for $det(E)$ and $det(\bar{E})$*

*is a morphism of modules* $\Phi_i : E_i \to \bar{E}_{i+m}$*, for each* $i$*, with* $Im(\Phi_i) \not\subset \bar{E}_{i+m-1}$*, preserving the diagrams that define the elliptic sheaves and their level structures.*

If $E$ and $\bar{E}$ are defined over $R$, then to give an isogeny, $\Phi : E \to \bar{E}$, of degree $m$ is equivalent to giving a morphism of $\tau$-sheaves $\phi : R\{\tau\} \to R\{\bar{\tau}\}$, such that if $r(\tau)$ is a monic polynomial with $deg_\tau(r(\tau)) = j$ then $deg_{\bar{\tau}}\phi(r(\tau)) = m + j$.

**Lemma 3.2.** Let $M$ and $N$ be vector bundles of rank $n$ over $\mathbb{P}^1_R$, and with $x$ a rational point of $\mathbb{P}^1$. If $f : M \to N$ is a morphism of modules such that its restriction to $k(x)$

$$f_{|k(x)\otimes R} : M_{|k(x)\otimes R} \to N_{|k(x)\otimes R}$$

is an isomorphism, then $f$ is injective.

*Proof.* Let assume us that $x$ is the rational point $0 \in \mathbb{P}^1$. We have the exact sequence:

$$0 \to K \to M \xrightarrow{f} N.$$

If we prove that $K_{(|\mathbb{P}_1 \setminus \{\infty\}) \otimes R} = 0$ then we conclude. Let

$$0 \to \hat{K} \to \hat{M} \xrightarrow{\hat{f}} \hat{N}$$

be the completion of the above exact sequence along the ideal $tR[t]$. By hypothesis, $f_{|k(x)\otimes R}$ is an isomorphism. One deduces that $\hat{f}$ is also an isomorphism and hence $\hat{K} = 0$, since

$$Spec_{\text{maximal}}(R[[t]]) = 0 \times Spec_{\text{maximal}}(R),$$

and in view of the Nakayama lemma. If we prove that the natural morphism $g : K \to \hat{K}$ is injective we conclude. By the Krull Theorem, if $g(a) = 0$ then there exists $1 + t \cdot q(t) \in R[t]$ such that $(1 + t \cdot q(t)) \cdot a = 0$. However, the homothety morphism given by $1 + t \cdot q(t)$ over $R[t]$ is injective and therefore it is also injective over $M$ because $M$ is locally free. Hence, $a = 0$.

□

**Lemma 3.3.** Assuming the above notations, if $\Phi$ is an isogeny of degree $m \leq n \cdot d$ between $(E, \iota_{I\infty}), (\bar{E}, \bar{\iota}_{I\infty})$ then $\Phi$ is injective and it is the only isogeny between

these elliptic sheaves with level structures. Moreover, there exists $r \in \mathbb{N}$, maximum $(r \leq n \cdot d)$, such that $\phi(R\{\tau\}) \subseteq R\{\bar{\tau}\} \cdot \bar{\tau}^r$.

*Proof.* We can assume that the elliptic sheaves are defined over an $\mathbb{F}_q$-algebra $R$. In this way, the injectivity is deduced from the above lemma. We denote by $I$ indistinctly the ideal within $\mathbb{F}_q[t]$ as the ideal sheaf within $\mathcal{O}_{\mathbb{P}^1}$

Let $\Phi'$ be another isogeny; $\Phi - \Phi'$ defines a morphism $E_0 \to I \cdot \bar{E}_{n \cdot d}$. Since $E_0$ is generated by its global sections, and since $deg(I \cdot \bar{E}_{n \cdot d})) = -n$, because $deg(I) = d+1$, we have $h^0(I \cdot \bar{E}_{n \cdot d}) = 0$. In this way we have that $\Phi = \Phi'$.

The last assertion of the lemma is evident.

□

We consider $|\mathbb{P}_1|_{nd}$, the subset of geometric points of $\mathbb{P}_1$, of degree less than or equal to $nd$. $\mathcal{E}^{I\infty}_{n,|\mathbb{P}^1|_{nd}}$ denotes the moduli scheme of elliptic sheaves of rank $n$ with level structures over $I$ and $\infty$ and with zero outside $|\mathbb{P}^1|_{nd}$.

**Lemma 3.4.** With the above notations, the set

$$[(E, \iota_{I\infty}), (\bar{E}, \bar{\iota}_{I\infty})] \in \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}},$$

such that there exists an isogeny of degree $m \leq n \cdot d$ between $(E, \iota_{I\infty})$ and $(\bar{E}, \bar{\iota}_{I\infty})$ with $r = 0$, is given by the correspondence $T(m) \subset \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$.

*Proof.* It is clear that a pair within $T(m)$ defines an isogeny of degree $m$ with the required properties. Moreover, the very lemma asserts that there only exists one isogeny of degree $m \leq nd$ between two elliptic sheaves with $I$-level structures. With this result, one deduces that if $(E, \iota_{I\infty})$ and $(E', \iota'_{I\infty})$ are subelliptic sheaves with level structures of $(\bar{E}, \bar{\iota}_{I\infty})$, by two different isogenies of degree $m$, then $(E, \iota_{I\infty})$ is not isomorphic to $(E', \iota'_{I\infty})$.

On the other hand, if $\Phi : (E, \iota_{I\infty}) \to (\bar{E}, \bar{\iota}_{I\infty})$ is an isogeny with $r = 0$ and degree $m$, then by the serpent lemma we have isomorphisms $(Id \times F)^*(\bar{E}_{i+m}/\Phi_i(E_i)) \simeq \bar{E}_{i+m}/\Phi_i(E_i)$, for each integer, $i$. Since the zeroes of the elliptic sheaves considered

are of degree $> nd$, we have

$$\bar{E}_{i+m}/\Phi_i(E_i) \simeq R[t]/J_1 \oplus \cdots \oplus R[t]/J_n,$$

where $J_1 \subseteq \cdots \subseteq J_n$ are ideals within $\mathbb{F}_q[t]$ coprime to $I$ with $\sum_{i=0}^n dim_{\mathbb{F}_q} A/J_i = m$. Here, we have assumed that $(E, \iota_{I\infty})$ and $(\bar{E}, \bar{\iota}_{I\infty})$ are defined over $R$.

□

**Corollary 3.5.** The subset of pairs $(e, \bar{e}) \in \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$ such that there exists an isogeny of degree $n \cdot d$ is given by the correspondence:

$$T(n\cdot d) + \Gamma(Fr) * T(n \cdot d - 1) + \cdots + \Gamma(Fr^{n\cdot d-1}) * T(1) + \Gamma(Fr^{n\cdot d}).$$

Here, $\Gamma(Fr^i)$ is given by the graph of the $q^i$-Frobenius morphism. $*$ denotes the product of correspondences.

*Proof.* The elliptic sheaf associated with the $\tau$-sheaf, $R\{\bar{\tau}\} \cdot \bar{\tau}^r$, is

$$[(Id \times F)^r]^* \bar{E}.$$

In view of the two last lemmas, the corollary is deduced bearing in mind that between $E_0$ and $[(Id \times F)^{nd+j}]^* \bar{E}_{-j}$ there is no injective morphism for $j > 0$, because $deg(E_0) = 0$ and $deg[((Id \times F)^{nd+j})^* \bar{E}_{-j}] = -j$. □

3.2. **Trivial correspondences.** In this section we shall prove that the correspondence of the above corollary 3.5 is trivial.

**Proposition 3.6.** Let $M$ be a vector bundle over $\mathbb{P}^1_R$ of rank $n$ and degree 0 where $h^0(M(-1)) = h^1(M(-1)) = 0$, and with an $I$-level structure $\iota_I$. Thus, we have that $H^0(\mathbb{P}^1_R, M)$ is a free $R$-module of rank $n$, and $M \simeq H^0(\mathbb{P}^1_R, M) \otimes \mathcal{O}_{\mathbb{P}_1}$.

*Proof.* If $x \in Spec(\mathbb{F}_q[t]/I)$ is a rational point, then $h^0(M(-x)) = h^1(M(-x)) = 0$. Bearing in mind the morphism given by the $x$-level structure $\iota_x : M \to (k(x) \otimes R)^n$, we obtain an isomorphism:

$$M/M(-x) \simeq (R[t]/\mathfrak{m}_x)^{\oplus n}.$$

Therefore, by taking global sections in the exact sequence of $\mathcal{O}_{\mathbb{P}_1} \otimes R$-modules

$$0 \to M(-x) \to M \to M/M(-x) \to 0$$

we conclude.

The argument is valid when $Spec(\mathbb{F}_q[t]/I)$ does not have rational points because $N$ is a $R$-free module if and only if $N \otimes \mathbb{F}_{q^d}$ is a $R \otimes \mathbb{F}_{q^d}$-free module.

$\square$

If $(\bar{M}, \bar{\iota}_I)$ is an $I$-level structure then we denote by $(\bar{M}(d), \bar{\iota}_I(d))$ the $I$-level structure over $\bar{M}(d)$ obtained from $\bar{\iota}_I$ by considering the natural inclusion $\bar{M} \subseteq \bar{M}(d)$. Recall that $\infty \notin |I|$.

**Lemma 3.7.** If $(\bar{M}, \bar{\iota}_I), (M, \iota_I)$ are level structures over $R$, where $M$ and $\bar{M}$ satisfy the conditions of the above proposition, then there exists a morphism of vector bundles, $h : M \to \bar{M}(d)$, whose diagram

$$\begin{tikzcd} M \arrow[r, "h"] \arrow[dr, "\iota_I"'] & \bar{M}(d) \arrow[d, "\bar{\iota}_I(d)"] \\ & (\beta_*(R[t]/I))^{\oplus n} \end{tikzcd}$$

is commutative. $h$ is said to be a morphism between $(M, \iota_I)$ and $(\bar{M}(d), \bar{\iota}_I(d))$.

*Proof.* By choosing a base, $\{s_1, \cdots, s_n\}$ for $H^0(\mathbb{P}^1_R, M)$.

$H^0(\iota_I) : H^0(M) \to (R[t]/I)^{\oplus n}$ has the associated matrix:

$$\Delta_0 + \Delta_1 t + \cdots + \Delta_d t^d,$$

where $\Delta_i$ are $n \times n$-matrices with entries in $R$.

We have that

$$H^0(\mathbb{P}^1_R, \bar{M}(d)) = \oplus_{i=0}^d H^0(\mathbb{P}^1_R, \bar{M}) \cdot t^i.$$

Moreover, bearing in mind that $deg(I) = d + 1$ we also have that

$$H^0(\mathbb{P}^1_R, \bar{M}(d)) \overset{H^0(\bar{\iota}_I)}{\simeq} (R[t]/I)^{\oplus n},$$

because $h^0(I \cdot \bar{M}(d)) = 0$. Thus, we have that $H^0(h)$ must satisfy

$$H^0(h) = A_0 + \cdots + A_d.t^d := (H^0(\bar{\iota}_I))^{-1} \cdot (\Delta_0 + \Delta_1 t + \cdots + \Delta_d t^d),$$

where $A_i$ are $n \times n$-matrices with entries in $R$. We conclude by bearing the above proposition in mind. $\square$

The same arguments of lemma 3.3 allow us to deduce that $h$ is unique.

By remark 1, if $E$ is an elliptic sheaf of rank $n$, then $E_0$ satisfies the conditions of proposition 3.6.

Let us consider the elliptic sheaves, defined over $R$, with $I$-level structures $(E, \iota_{I\infty}), (\bar{E}, \bar{\iota}_{I\infty})$ and $\infty$-level structures for $det(E)$ and $det(\bar{E})$.

**Lemma 3.8.** Let $h$ be the morphism between vector bundles with level structures $h : (E_0, \iota_{0,I}) \to (\bar{E}_0(d), \bar{\iota}_{0,I}(d))$ given in lemma 3.7, and let $\iota_\infty$, $\bar{\iota}_\infty$ be level structures at $\infty$ over $det(E)$ and $det(\bar{E})$, respectively. Therefore, there exists an isogeny of degree $n \cdot d$, $\Phi : (E, \iota_{I\infty}) \to (\bar{E}, \bar{\iota}_{I\infty})$ with $\Phi_0 = h$ if and only if $det(h)$ is a morphism for the level structures $(det(E_0), \iota_\infty)$, $(det(\bar{E}_0(d)), \bar{\iota}_\infty)$, (i.e: $\bar{\iota}_\infty \cdot det(h) = \iota_\infty$), and the morphism of $R[t]$-modules given by $h$, $h_A : R\{\tau\} \to R\{\bar{\tau}\}$ satisfies $deg_{\bar{\tau}}(h(1)) \leq n - 2 + nd, \cdots, deg_{\bar{\tau}}(h(\tau^{n-2})) \leq n - 2 + nd$.

*Proof.* The direct way is trivial.

We prove the converse. Since $h$ is a morphism for the $I$-level structures $(E_0, \iota_{0,I})$ and $(\bar{E}_0(d), \bar{\iota}_{0,I}(d))$, we have that

$$h_A(\tau) - \bar{\tau} \cdot h_A(1), \cdots, h_A(\tau^{n-1}) - \bar{\tau} \cdot h_A(\tau^{n-2}) \in I \cdot R\{\bar{\tau}\}.$$

Moreover, since $deg(I) = d + 1$, if $r(\bar{\tau}) \neq 0 \in R\{\bar{\tau}\} \cdot I$ then $deg_{\bar{\tau}}(r(\bar{\tau})) \geq n(d + 1)$. Thus, by the hypothesis of the lemma we deduce that

$$h_A(\tau) - \bar{\tau} \cdot h_A(1) = \cdots = h_A(\tau^{n-1}) - \bar{\tau} \cdot h(\tau^{n-2}) = 0.$$

Therefore, $h_A(\tau^i) = \bar{\tau}^j \cdot h_A(\tau^k)$ for $j + k = i$ and $1 \leq i \leq n - 1$.

Now, we prove the equalities

$$(*) \quad deg_{\bar{\tau}}(h_A(1)) = nd, deg_{\bar{\tau}}(h_A(\tau)) = nd + 1, \cdots, deg_{\bar{\tau}}(h_A(\tau^{n-1})) = nd + n - 1.$$

We consider the determinant elliptic sheaves $det(E)$ and $det(\bar{E})$ and their $\tau$-sheaves $R\{\tau_{det}\}$ and $R\{\bar{\tau}_{det})\}$, respectively. We have that

$$[det(h_A) \cdot \tau_{det} - \bar{\tau}_{det} \cdot det(h_A)](1 \wedge \tau \wedge \cdots \wedge \tau^{n-2} \wedge \tau^{n-1})$$

is an element of $R\{\tau_{det}\}$ of degree $\leq nd+1$. However, by hypothesis $det(h)$ is a morphism for $\infty$-level structures for elliptic sheaves and therefore this element is of degree $\leq nd$.

Because $h_A(\tau^i) = \bar{\tau} \cdot h_A(\tau^{i-1})$, for $1 \leq i \leq n-1$, the above element of $R\{\tau_{det}\}$ is equal to

$$h_A(\tau) \wedge \cdots \wedge h_A(\tau^{n-1}) \wedge [h_A \cdot \tau - \bar{\tau} \cdot h_A](\tau^{n-1}).$$

Since $deg_{\bar{\tau}}(h_A(\tau^{n-1})) \leq nd+n-1$ and $h_A(\tau^i) = \bar{\tau}^i \cdot h_A(1)$ for $1 \leq i \leq n-1$, we have the inequalities

$$deg_{\bar{\tau}}(h_A(1)) \leq nd, deg_{\bar{\tau}}(h_A(\tau)) \leq nd+1, \cdots, deg_{\bar{\tau}}(h_A(\tau^{n-1})) \leq nd+n-1.$$

But $\bar{E}_0(d)/h(E_0)$ is not concentrated in $\infty$, because $\bar{\iota}_\infty \cdot det(h) = \iota_\infty$ and $\bar{\iota}_\infty$, $\iota_\infty$ are surjective morphisms, and hence one deduces the equalities (*).

Using remark 2, since

$$h_A(\tau) \wedge \cdots \wedge h_A(\tau^{n-1}) \wedge [h_A \cdot \tau - \bar{\tau} \cdot h_A](\tau^{n-1})$$

is an element of $R\{\tau_{det}\}$ of degree $\leq nd$, we have that

$$deg_{\bar{\tau}}[h_A \cdot \tau - \bar{\tau} \cdot h_A](\tau^{n-1}) \leq nd+n-1,$$

and we conclude that $[h_A \cdot \tau - \bar{\tau} \cdot h_A](\tau^{n-1}) = 0$ because

$$[h_A \cdot \tau - \bar{\tau} \cdot h_A](\tau^{n-1}) \in R\{\bar{\tau}\} \cdot I$$

and $deg(I) = d+1$. Thus, $h_A : R\{\tau\} \to R\{\bar{\tau}\}$ is an isogeny of degree $nd$.

□

**Lemma 3.9.** Let $X = Spec(A)$ be a smooth, noetherian scheme of dimension $2n$. Let $Z_1 + \cdots + Z_r$ be an $n$-cycle in $X$ such that $Z_i$ are different irreducible closed subschemes of dimension $n$ in $X$. If the closed subscheme $Z := Z_1 \cup \cdots \cup Z_r$ is given by an ideal generated by $n$ elements $a_1, \cdots, a_n \in A$, then the $n$-cycle $Z_1 + \cdots + Z_r$ is rationally equivalent to 0.

*Proof.* Let $\mathfrak{I}$ be an ideal within $A$. We denote by $Z(\mathfrak{I})$ the cycle associated in $X$ with the closed subscheme given by $\mathfrak{I}$. The prime ideal within $A$ given by $Z_i$ is

denoted by $P_i$. Thus,

$$Z(P_1 \cap \cdots \cap P_r) = Z_1 + \cdots + Z_r.$$

Let us consider the ideal $(a_2, \cdots, a_n)$ within $A$ generated by $a_2, \cdots, a_n$ and let $Q_1 \cap \cdots \cap Q_h$ be a minimal primary decomposition of this ideal. If $Y_1, \cdots, Y_k, (k \leq h)$ are the irreducible components of the closed subscheme within $X$ given by $(a_2, \cdots, a_n)$, then $\dim Y_j \geq n+1$. We may assume, reordering the indices, that $Z(Q_1) = Y_1, \cdots, Z(Q_k) = Y_k$.

By taking the localisation by $P_i$, one obtains

$$(A/Q_1 \cap \cdots \cap Q_h)_{P_i},$$

which is a local ring of dimension 1 because $Y_1, \cdots, Y_k$ has $\dim Y_j \geq n+1$. From the equality of rings

$$A/Q_1 \cap \cdots \cap Q_h + (a_1) = A/P_1 \cap \cdots \cap P_r$$

one obtains

$$(A/Q_1 \cap \cdots \cap Q_h + (a_1))_{P_i} = (A/P_i)_{P_i}.$$

Therefore, $(A/Q_1 \cap \cdots \cap Q_h)_{P_i}$ is principal and hence an integral domain, and therefore there exists a unique $Q_{l_i}, (l_i \leq k)$ with $Q_{l_i} \subset P_i$. If we denote by $P_{j_1}, \cdots, P_{j_{m_j}}$ the $P_i$'s with $Q_j \subset P_{j_1}, \cdots, Q_j \subset P_{j_{m_j}}, (j \leq k)$, then within the $n+1$-dimensional scheme, $Z(Q_j) = Y_j$

$$Z_{j_1} + \cdots + Z_{j_{m_j}}$$

is given by the zero locus of $a_1$, which proves that $Z_{j_1} + \cdots + Z_{j_{m_j}}$ is rationally equivalent to 0 on $X$.

We conclude because $Z_1 + \cdots + Z_r = \sum_{j=1}^{k} Z_{j_1} + \cdots + Z_{j_{m_j}}$.

□

**Theorem 3.10.** The correspondence

$$T_I^n := T(n \cdot d) + \Gamma(Fr) * T(n \cdot d - 1) + \cdots + \Gamma(Fr^{n \cdot d - 1}) * T(1) + \Gamma(Fr^{n \cdot d})$$

is trivial(= rationally equivalent to 0 as an $n$-cycle within $\mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$).

*Proof.* Bearing in mind corollary 3.5 and lemma 3.8, this correspondence is given by the zero locus of $n$ regular functions of $\mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$.

We are within the hypothesis of the above lemma because the projection over the first entry, $T(r) \to \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$ is an étale morphism and therefore $T(r)$ is smooth. Moreover, because of lemma 3.3 if $i \neq j$ then:

$$\Gamma(Fr^i) * T(n \cdot d - i) \cap \Gamma(Fr^j) * T(n \cdot d - j) = \emptyset,$$

with $i, j \leq n \cdot d$.

□

3.3. **Some explicit calculations.** One can make explicit calculations by using the anti-equivalence between elliptic sheaves and Drinfeld modules (c.f:[Dr2], [Mu]) and by using the explicit calculation of the global sections "$s$", ([Al1], remark 3.1), in terms of the $I$-torsion elements of the Drinfeld modules. For $n = 1$, calculations are made in [An2] and in the spirit of this work in [Al3]: (example 2, page 21) and in [Al2], 3.2.

We begin with the following example.

*Example* 3.11. $n = 1$, $I = p(t)\mathbb{F}_q[t]$, $p(t) = (t - a_1) \cdots (t - a_{d+1})$, with $a_i \neq a_j$ for $i \neq j$ and $a_i \in \mathbb{F}_q$. Let $(L_j, i_j, \tau)$ be the rank-one elliptic sheaf defined over the Carlitz's cyclotomic ring $K_1^{I\infty} = \mathbb{F}_q(\bar{t})[\delta]$, with $\delta$ an element of a closed field of $\mathbb{F}_q(t)$ verifying:

$$\phi_{p(t)}(\delta) = \delta^{q^{d+1}} + \cdots + c_1\delta^q + p(\bar{t})\delta = 0,$$

where $\phi$ is the Drinfeld module $\phi_t = \tau + \bar{t}$, (remark 3 of section 2.2).

Let us consider the $I\infty$-level structure, $\iota_{I\infty}$, for $(L_j, i_j, \tau)$. We have

$$\iota_I : L_0 \to K_1^{I\infty}[t]/p(t)$$

given by $\iota_{I\infty}(s) = m_1\delta_1 \frac{p(t)}{t-\alpha_1} + \cdots + m_{d+1}\delta_{d+1} \frac{p(t)}{t-\alpha_{d+1}}$ and

$$\iota_\infty : L_0 \to K_1^{I\infty}[t^{-1}]/t^{-1},$$

with $\iota_\infty(s)=1$. Here $L_0=s\cdot\mathcal{O}_{\mathbb{P}^1}\otimes K_1^{I\infty}$, $\phi_{\frac{p(t)}{t-a_j}}(\delta)=\delta_j$ and the $m_j$ are obtained from the equality

$$\frac{1}{p(t)}=\frac{m_1}{t-a_1}+\cdots+\frac{m_{d+1}}{t-a_{d+1}}.$$

We shall obtain the element of $K_1^{I\infty}\otimes K_1^{I\infty}$ whose divisor is

$$T(d)+\Gamma(Fr)*T(d-1)+\cdots+\Gamma(Fr^{d-1})*T(1)+\Gamma(Fr^d).$$

Let $\pi_1$ and $\pi_2$ be the natural projections

$$Spec(K_1^{I\infty}\otimes K_1^{I\infty})\to Spec(K_1^{I\infty}).$$

The morphism, $h$, of lemma 3.7 applied to the rank-one line bundles with a $I$-level structure, $\pi_1^*(L_0,\iota_I)$ and $\pi_2^*(L_0(d),\iota_I)$, is given by:

$$h(\pi_1^*s)=[m_1\frac{\delta_1\otimes 1}{1\otimes\delta_1}\frac{p(t)}{t-\alpha_1}+\cdots+m_{d+1}\frac{\delta_{d+1}\otimes 1}{1\otimes\delta_{d+1}}\frac{p(t)}{t-\alpha_{d+1}}]\pi_2^*s.$$

By lemma 3.8, one must impose on $h$ that:

$$\pi_2^*\iota_\infty(h(\pi_1^*s))=\pi_1^*\iota_{I\infty}(\pi_1^*s)$$

by the definition of $\infty$-level structures

$$\pi_2^*\iota_\infty(h(\pi_1^*s))=m_1\frac{\delta_1\otimes 1}{1\otimes\delta_1}+\cdots+m_{d+1}\frac{\delta_{d+1}\otimes 1}{1\otimes\delta_{d+1}}$$

which is the leader coefficient of the polynomial

$$m_1\frac{\delta_1\otimes 1}{1\otimes\delta_1}\frac{p(t)}{t-a_1}+\cdots+m_{d+1}\frac{\delta_{d+1}\otimes 1}{1\otimes\delta_{d+1}}\frac{p(t)}{t-a_{d+1}}.$$

Since $\iota_{I\infty}(s)=1$, the element sought is

$$m_1\frac{\delta_1\otimes 1}{1\otimes\delta_1}+\cdots+m_{d+1}\frac{\delta_{d+1}\otimes 1}{1\otimes\delta_{d+1}}-1.$$

*Example* 3.12. Now, we follow with the easiest anabelian case.

$n=2$, $I=t\mathbb{F}_q[t]$. Let $\phi_t=a\sigma^2+b\sigma+\bar{t}$ be a Drinfeld module of rank two defined over the ring

$$B_2^{I\infty}=(\mathbb{F}_q[a,a^{1/1-q},b,\bar{t},r(\bar{t})^{-1}]/a+b+\bar{t}-1)[\Gamma],$$

with $\phi_t(1)=0$, $\Gamma^q-\Gamma\neq 0$ and $\phi_t(\Gamma)=0$. $r(t)$ is the product of the monic polynomials of degree less than or equal to 2. Let $(M_j,i_j,\tau)$ be the rank-two

elliptic sheaf associated with $\phi$, and let $\iota_{I\infty}$ be an $I\infty$-level structure for $(M_j, i_j, \tau)$ given by:

$$\iota_I : M_0 \to (B_2^{I\infty}[t]/t)^{\oplus 2} \simeq (B_2^{I\infty})^{\oplus 2},$$

given by $\iota_{I\infty}(s) = (1, \Gamma)$ and $\iota_{I\infty}(\tau s) = (1, \Gamma^q)$. Recall that:

$$M_0 = s \cdot (\mathcal{O}_{\mathbb{P}^1} \otimes B_2^{I\infty}) \oplus \tau s \cdot (\mathcal{O}_{\mathbb{P}^1} \otimes B_2^{I\infty}).$$

The $\infty$-level structure

$$\iota_\infty : Det(M_0) \to B_2^{I\infty}[t^{-1}]/t^{-1}$$

is given by $\iota_\infty(s \wedge \tau s) = a$.

Let $\pi_1$ and $\pi_2$ be the natural projections:

$$Spec(B_2^{I\infty} \otimes B_2^{I\infty}) \to Spec(B_2^{I\infty}).$$

The morphism, $h$, of lemma 3.7 applied to the rank-two vector bundles with a $I$-level structure, $\pi_1^*(M_0, \iota_I)$ and $\pi_2^*(M_0, \iota_I)$, (here $d = 0$) is given by the matrix product

$$D := \begin{pmatrix} 1 & 1 \\ 1 \otimes \Gamma & 1 \otimes \Gamma^q \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 \\ \Gamma \otimes 1 & \Gamma^q \otimes 1 \end{pmatrix}.$$

By lemma 3.8, one must impose on $h$, that $deg(h_A(1)) = 0$, where

$$h_A : B_2^{I\infty} \otimes B_2^{I\infty}\{\tau\} \to B_2^{I\infty} \otimes B_2^{I\infty}\{\tau\}$$

is the restriction of $h$ to $\mathbb{P}^1 \setminus \{\infty\}$. By considering the second entry of $D \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, this condition is:

$$* := (1 \otimes \Gamma^q - 1 \otimes \Gamma)^{-1}(\Gamma \otimes 1 - 1 \otimes \Gamma) = 0.$$

We must now impose that:

$$\pi_2^* \iota_\infty(det(h)(\pi_1^*(s \wedge \tau s))) = \pi_1^* \iota_{I\infty}(\pi_1^*(s \wedge \tau s)).$$

Since, $\iota_\infty(s \wedge \tau s) = a$, we have:

$$\pi_2^* \iota_\infty(det(h)(\pi_1^*(s \wedge \tau s))) = Det(D) \cdot (1 \otimes a),$$

and

$$\pi_1^* \iota_{I\infty}(\pi_1^*(s \wedge \tau s)) = a \otimes 1.$$

Thus we obtain the element:

$$** := (\Gamma^q - \Gamma) \otimes (\Gamma^q - \Gamma)^{-1} - a \otimes a^{-1}$$

The final result is that the diagonal subscheme of $Spec(B_2^{I\infty} \otimes B_2^{I\infty})$ is the zero locus of the ideal generated by $*$ and $**$.

## 4. The additive case: $n = 2$ (anhilators for cusp forms)

In this section, we shall follow the notation setout in the introduction. The set of cusps is $\overline{\mathcal{E}}(I\infty) \setminus \mathcal{E}(I\infty)$. We denote by $\mathcal{C}^0(I\infty)$ the divisor class group on $\overline{\mathcal{E}}(I\infty)$ whose support lies among the cusps. As in the introduction we follow the notation and results of [GR]. For the definition and study of cusps forms, readers are referred to the works of Gekeler, Goss or the Habilatationshrift of Gebhard Böckle.

We now prove a Lemma which is the counterpart for $n = 2$ for Stickelberger's Theorem.

**Lemma 4.1.** $T(2d)+T(2d-1)+\cdots+T(1)+\Gamma(Id)$ annihilates the group $Pic(\mathcal{E}(I\infty))$.

*Proof.* This lemma is proved by using theorem 6.1, and the fact that the divisor group, $\mathcal{D}^0(\mathcal{E}(I\infty)$, of the affine curve, $\mathcal{E}(I\infty)$ defined over $K_I^\infty$ is a subgroup of the group of Weil divisors of $\mathcal{E}^{I\infty}_{2,|\mathbb{P}_1|_{2d}}$. Recall that $\mathcal{E}^{I\infty}_{2,|\mathbb{P}_1|_{2d}}$ is a smooth variety of dimension 2. □

Note that the Hecke correspondences operate over the cusps. Thus, the above correspondence gives a group endomorphism, $\mathcal{C}^0(I\infty) \to \mathcal{C}^0(I\infty)$. We denote by $\overline{S}_2(d)$, $S_2(d)$ and $S'_2(d)$ the group endomorphisms given by,

$$T(2d) + T(2d-1) + \cdots + T(1) + \Gamma(Id)$$

over the groups $Pic^0(\overline{\mathcal{E}}(I\infty))$, $Pic(\mathcal{E}(I\infty))$ and $\mathcal{C}^0(I\infty)$, respectively.

Let us consider $j^*$, the pull back of the line bundles over $\overline{\mathcal{E}}(I\infty)$ by the natural inclusion:

$$j : \mathcal{E}(I\infty) \hookrightarrow \overline{\mathcal{E}}(I\infty).$$

We assume that:

$$j^* : Pic^0(\overline{\mathcal{E}}(I\infty)) \to Pic(\mathcal{E}(I\infty))$$

is surjective. For example if in the cusps points $\overline{\mathcal{E}}(I\infty)\backslash\mathcal{E}(I\infty)$ there exists a rational point over $K_I^\infty$. If this does not occur then it suffices to change $Pic(\mathcal{E}(I\infty))$ by $Pic^0(\mathcal{E}(I\infty))$.

**Lemma 4.2.** If $Pic(\mathcal{E}(I\infty))$ is an infinity group, then $Ker(\overline{S}_2(d))$ also has infinity elements.

*Proof.* If $Coker(S)$ is not finite then we finish the proof, since $\mathcal{C}^0(I\infty)$ is a finite generated group. Thus, we can assume that $ker(S'{}_2(d))$ and $coker(S'{}_2(d))$ are finite groups.

From the serpent lemma applied to the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^0(I\infty) \arrow[r] \arrow[d, "S'_2(d)"] & Pic^0(\overline{\mathcal{E}}(I\infty)) \arrow[r, "j^*"] \arrow[d, "\overline{S}_2(d)"] & Pic(\mathcal{E}(I\infty)) \arrow[r, "j"] \arrow[d, "S_2(d)"] & 0 \\
0 \arrow[r] & \mathcal{C}^0(I\infty) \arrow[r] & Pic^0(\overline{\mathcal{E}}(I\infty)) \arrow[r, "j^*"] & Pic(\mathcal{E}(I\infty)) \arrow[r] & 0
\end{tikzcd}$$

one obtains an exact sequence:

$$Ker(S'_2(d)) \to Ker(\overline{S}_2(d)) \to Pic(\mathcal{E}(I\infty)) \stackrel{\delta}{\to} Coker(S'_2(d)).$$

We conclude since $Ker(S'_2(d))$, and $Coker(S'_2(d))$ are finite groups and because by hypothesis, $Pic(\mathcal{E}(I\infty))$ is an infinity group. $\square$

**Theorem 4.3.** If the the cardinal of the group $Pic(\mathcal{E}(I\infty))$ is $\infty$, then there exists a cusp form for $\Gamma_{I\infty}$ that is annihilated by $\tilde{T}(2d) + \tilde{T}(2d-1) + \cdots + \tilde{T}(1) + Id$.

*Proof.* We denote by $J$ the Jacobian of the curve $\overline{\mathcal{E}}(I\infty)$ over $K_I^\infty$. Thus, the correspondence $\overline{S}_2(d)$ gives an endomorphism of this Jacobian. By the last lemma, this endomorphism can not be an isogeny. Accordingly, the morphism induced over the tangent space $T_e(J)$ of $J$ over the zero element,

$$\tilde{S}_2(d) : T_e(J) \to T_e(J)$$

is not injective. We conclude because the tangent space, $T_e(J)$, is the dual space of the space of 1-holomorphic differential forms, $H^0(\overline{\mathcal{E}}(I\infty), \Omega_{\overline{\mathcal{E}}(I\infty)})$, and the space of cusp forms is identified with

$$H^0(\overline{M}_{\Gamma_{I\infty}}, \Omega_{\overline{M}_{\Gamma_{I\infty}}}) = H^0(\overline{\mathcal{E}}(I\infty), \Omega_{\overline{\mathcal{E}}(I\infty)}) \otimes_{K_I^\infty} C.$$

□

## 5. Ideal class group anhilators for the cyclotomic function fields

We consider $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}} = Spec(\mathcal{B}^{I\infty}_1)$. The construction of $\mathcal{B}^{I\infty}_1$ is detailed in section 2.2, remark 3, and is essentially as follows: Let $((L_j, i_j, \tau), \iota_{I\infty})$ be an element of $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$. To construct $\mathcal{B}^{I\infty}_1$, we can fix a global section $s$ of $L_0$ such that $t \cdot s = \lambda \cdot s + \tau s$, and $\iota_\infty(s) = 1$. Hence, $Spec(\mathcal{B}^{I\infty}_1)$ represents the pairs $(\phi, \iota_I)$, with $\phi$ a rank 1-normalized Drinfeld module and $\iota_I$ an $I$-level structure for $\phi$. $\mathcal{B}^{I\infty}_1$ is considered in [An1] and [C] and is obtained from the $I$-torsion elements of a normalized Drinfeld module. The "zero" morphism $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}} \to Spec(\mathbb{F}_q[t])$ gives a Galois extension $K^\infty_I/\mathbb{F}_q(t)$ of group $(\mathbb{F}_q[t]/I)^\times$. We denote by $Y^\infty_I$ the proper smooth curve associated with $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$.

We consider the Hecke correspondence:

$$T(J_1, \cdots, J_n) \subset \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}},$$

which is of degree $d(J_1, \cdots, J_n)$ over the second component. Let $J$ be the product of ideals:

$$\prod_{i=1}^{n} J_i := J.$$

$T(J)$ denotes the Hecke correspondence on $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$ given by $J$.

There exists an action, $T(J_1, \cdots, J_n)^*$ and $T(J)^*$, of these correspondences over the functors, $\underline{Pic}(\mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}})$ and $\underline{Pic}^0(Y^\infty_I)$, respectively. These functors are defined over the category of $\mathbb{F}_q$-schemes. Recall that the projections $\bar{\pi}_1$ and $\bar{\pi}_2$

$$T(J_1, \cdots, J_n) \to \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$$

are étale. In this way it is possible to define $T(J_1, \cdots, J_n)^* (:= (\bar{\pi}_2)_* \cdot \bar{\pi}_1^*)$.

*Remark* 4. Let us consider the morphism $det : \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \to \mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$. We set

$$T(J_1, \cdots, J_n)^* det^*[D] = d(J_1, \cdots, J_n) det^* T(J)^*[D]$$

$[D]$ is the class of a divisor $D$ on $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$.

This result is proved bearing in mind the projection formula for the rational equivalence of cycles; that $\bar{\pi}_2$ is an étale morphism of degree $d(J_1, \cdots, J_n)$, and

that given $e := (E, \iota_{I\infty}) \in \mathcal{E}_n^{I\infty}$ we have:

$$\bar{\pi}_2[T(J_1, \cdots, J_n) \cap (det^{-1}(det(e)) \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}})] = det^{-1}(T(J)^* det(e)),$$

$$T(J_1, \cdots, J_n) \cap (det^{-1}(det(e)) \times \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}) = \bar{\pi}_2^{-1}[det^{-1}(T(J)^* det(e))].$$

**Lemma 5.1.** The correspondence

$$D_I^n := \sum_{i=0}^{nd-i} \Gamma(Frob^i) * [\sum_{\substack{J \subset \mathbb{F}_q[t] \\ J+I=\mathbb{F}_q[t], deg(J)=i}} (\sum_{\substack{J_1 \subseteq \cdots \subseteq J_n \\ \prod_{k=1}^n J_k = J}} d(J_1, \cdots, J_n))T(J)]$$

is trivial on $Y_I^\infty \times Y_I^\infty$ up to vertical and horizontal correspondences.

*Proof.* It suffices to consider a curve $Z \to \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}}$ such that the morphism composition:

$$g : Z \to \mathcal{E}^{I\infty}_{n,|\mathbb{P}_1|_{nd}} \overset{det}{\to} \mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$$

is not constant. By the above remark, $(det)^* \cdot (D_I^n)^* = (T_I^n)^* \cdot (det)^*$. Since $T_I^n$ is rationally equivalent to zero, we have that $(g)_*(g)^* \cdot (D_I^n)^*$ is trivial on $Y_I^\infty \times Y_I^\infty$ up to vertical and horizontal correspondences, but by the projection formula

$$(g)_*(g)^* \cdot (D_I^n)^* = m \cdot D_I^n,$$

with $m \in \mathbb{N}$. We conclude, since the ring of correspondences, modulo horizontal and vertical ones, is without $\mathbb{Z}$-torsion.

□

By using the Euler product of lemma 2.6 one could find another proof of this lemma.

We consider $J = q(t)\mathbb{F}_q[t]$ with $q(t)$ monic. $T(J)$ is given by the graphic, $\Gamma(q(t))$, of the automorphism of $\mathcal{E}^{I\infty}_{1,|\mathbb{P}_1|_{nd}}$ obtained from the action of $q(t) \in (\mathbb{F}_q[t]/I)^\times$. Recall that to obtain $\mathcal{B}_1^{I\infty}$ we have fixed a global section $s$ of $L_0$ such that $t \cdot s = \lambda \cdot s + \tau s$, and $\iota_\infty(s) = 1$. In this way, $T(J) = \Gamma(q(t))$. By section 2.3, if we set $J_i = q_i(t)\mathbb{F}_q[t]$, then:

$$\varphi(q(t), n) := \sum_{\substack{J_1 \subseteq \cdots \subseteq J_n \\ \prod_{k=1}^n J_k = J}} d(J_1, \cdots, J_n))$$

is the number of submodules $N \subseteq \mathbb{F}_q[t]^{\oplus n}$ such that:

$$\mathbb{F}_q[t]^{\oplus n}/N \simeq \mathbb{F}_q[t]/q_1(t) \oplus \cdots \oplus \mathbb{F}_q[t]/q_n(t),$$

with the product of the invariant factors $q_1(t) \cdots q_r(t)$ equal to $q(t)$. Therefore, if we consider $p(t)\mathbb{F}_q[t] = I$, then:

**Corollary 5.2.** The correspondence

$$\sum_{i=0}^{nd} [\Gamma(Frob^{nd-i}) * ( \sum_{\substack{q(t) \text{ monic} \in \mathbb{F}_q[t] \\ (p(t),q(t))=1, deg(q(t))=i}} \varphi(q(t), n) \cdot \Gamma(q(t))]$$

is trivial on $Y_I^\infty \times Y_I^\infty$ up to vertical and horizontal correspondences.

*Example* 5.3. We can check this result for $n = 2$ and $p(t) = t(t-1)$. Let $K_{t(t-1)}^\infty/\mathbb{F}_q(t)$ be the Galois extension of group $(\mathbb{F}_q[t]/t(t-1))^\times$.

One has that $\varphi(t-\alpha, 2) = q+1$, $\varphi((t-\alpha)^2, 2) = q^2+q+1$, $\varphi((t-\alpha)(t-\beta), 2) = q^2 + 2q + 1$, and $\varphi(t^2 + at + b, 2) = q^2 + 1$ with $t^2 + at + b \in \mathbb{F}_q[t]$ an irreducible polynomial and $\alpha, \beta \in \mathbb{F}_q$, $\alpha \neq \beta$. Thus

$$(*) \quad \sum_{i=0}^{2} [\Gamma(Frob^{2-i}) * ( \sum_{\substack{q(t) \text{ monic} \in \mathbb{F}_q[t] \\ (t(t-1),q(t))=1, deg(q(t))=i}} \varphi(q(t), 2) \cdot \Gamma(q(t))]$$

is

$$\begin{gathered}
\Gamma(Frob^2) + (q+1) \sum_{\alpha \neq 0,1} \Gamma(Frob) * \Gamma(t-\alpha) + \\
+(q^2+2q+1) \sum_{\substack{\{\alpha,\beta\} \subset \mathbb{F}_q \\ \alpha,\beta \neq 0,1 \\ \alpha \neq \beta}} \Gamma((t-\alpha)(t-\beta)) + \\
+(q^2+q+1) \sum_{\alpha \neq 0,1} \Gamma((t-\alpha)^2) + \\
+(q^2+1) \sum_{\substack{a,b \in \mathbb{F}_q \\ t^2+at+b, \text{irreducible}}} \Gamma(t^2+at+b),
\end{gathered}$$

and grouping terms, we have:

$$\begin{gathered}
\Gamma(Frob) * [\Gamma(Frob) + \sum_{\alpha \neq 0,1} \Gamma(t-\alpha)] + \\
+q[\sum_{\alpha \neq 0,1} \Gamma(t-\alpha)] * [\Gamma(Frob) + \sum_{\alpha \neq 0,1} \Gamma(t-\alpha)] +
\end{gathered}$$

$$+(q^2+1)[\sum_{\substack{\{\alpha,\beta\}\subset\mathbb{F}_q \\ \alpha,\beta\neq 0,1 \\ \alpha\neq\beta}} \Gamma((t-\alpha)(t-\beta)) + \sum_{\alpha\neq 0,1}\Gamma((t-\alpha)^2) + \sum \Gamma(t^2+at+b)].$$

Now, bearing in mind that the last sumand is:

$$(q^2+1)(\sum_{g\in(\mathbb{F}_q[t]/t(t-1))^\times}\Gamma(g)),$$

which is a trivial correspondence, we conclude that (*) is also trivial because the correspondence

$$\Gamma(Frob) + \sum_{\alpha\in\mathbb{F}_q\setminus\{0,1\}}\Gamma(t-\alpha)$$

is trivial on $K^\infty_{t(t-1)}\otimes K^\infty_{t(t-1)}$ c.f. [C].

## 6. The above results without $\infty$ level structures

With minor changes in the above results one can obtain similar results but over the modular varieties, $\mathcal{E}^I_n$. The results obtained match, for $n=1$, the classical Stickelberger's theorem over $\mathbb{Z}$. (c.f [Gr1], [Gr2]).

To obtain these results it suffices to replace in lemma 3.8 the condition imposed on $h$ to be a morphism of $\infty$-level structures, by the condition:

$$deg_{\bar\tau}(h_A(\tau^n)-\bar\tau\cdot h_A(\tau^{n-1})\leq n-1+nd.$$

And now in corollary 3.5 one allows pairs, $[(E,\iota_I),(\bar E,\bar\iota_I)]$, given by an isogeny for $I$-level structures, $\Phi:(E,\iota_I)\to(\bar E,\bar\iota_I)$, such that $\infty$ can be within $supp(\bar E/\Phi(E))$. Thus, one obtains:

**Theorem 6.1.** The correspondence

$T(n\cdot d)+[\Gamma(Fr)+\Gamma(Id)]*T(n\cdot d-1)+\cdots+[\Gamma(Fr^{n\cdot d-1})+\cdots+\Gamma(Id)]*T(1)+$
$+[\Gamma(Fr^{n\cdot d})+\Gamma(Fr^{n\cdot d-1})+\cdots+\Gamma(Fr)+\Gamma(Id)]$

is trivial(= rationally equivalent to 0 as an $n$-cycle) within $\mathcal{E}^I_{n,|\mathbb{P}_1|_{nd}}\times\mathcal{E}^I_{n,|\mathbb{P}_1|_{nd}}$.

From the last theorem one has, for $n=2$:

**Lemma 6.2.** The correspondence $T(2d)+2T(2d-1)+\cdots+2dT(1)+(2d+1)\Gamma(Id))$ annihilates the group $Pic(\mathcal{E}(I))$.

and

**Theorem 6.3.** If the the cardinal of the group $Pic(\mathcal{E}(I))$ is $\infty$, then there exists a cusp form for $\Gamma_I$ that is annihilated by

$$\tilde{T}(2d) + 2\tilde{T}(2d-1) + \cdots + 2d\tilde{T}(1) + (2d+1)\Gamma(Id)).$$

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